

PEAK-LOAD PRICING ON A NETWORK

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ABSTRACT. In this paper we analyze peak-load pricing in the presence of network constraints. In our setup, firms facing fluctuating demand decide on the size and location of production facilities. They make production decisions constrained by the invested capacities, taking into account that market prices reflect scarce transmission capacities. We state general conditions for existence and uniqueness of the market equilibrium and provide a characterization of equilibrium investment and production. The presented analysis covers the cases of perfect competition and monopoly—the case of strategic firms is approximated by a conjectural variations approach. Our result is a prerequisite for analyzing regulatory policy options with computational multilevel equilibrium models, since uniqueness of the equilibrium at lower levels is of key importance when solving these models. Thus, our paper contributes to an evolving strand of literature that analyzes regulatory policy based on computational multilevel equilibrium models and aims at taking into account individual objectives of various agents, among them not only generators and customers but also, e.g., the regulator deciding on network expansion.

1. INTRODUCTION

The peak-load pricing literature analyzes investment incentives in industries where demand is fluctuating and storability of the output is limited; see Crew et al. (1995) for an overview. In such an environment firms will find it optimal to invest in a differentiated portfolio of base- and peak-load technologies. For the case of perfectly competitive markets, the unique equilibrium of this game is welfare optimal, i.e., firms take the right investment and production decisions. The approach of peak-load pricing is currently extensively used to analyze electricity markets, e.g., by Murphy and Smeers (2005) or Joskow and Tirole (2007), and many others.

The scope of this paper is to extend existence and uniqueness results of the peak-load pricing literature to the case where producers and consumers interact on a network. This is an important contribution to the literature on liberalized electricity markets, where typically private firms decide on investment and production, guided by incentives from spot market trading. In such an environment an adequate model of peak-load pricing on a network must account for the network constraints that the agents face at the spot markets whenever they are reflected in the spot market prices. The ability to establish a unique solution of this game is a prerequisite to meaningfully analyze complementary decisions taken by other agents—such as the regulator’s decisions on network expansion or the regulatory framework itself; see e.g., the analysis in Grimm et al. (2015).

In this paper we propose a framework that captures trading at spot markets, where market prices reflect scarce network capacities. Demand at each node is fluctuating. We analyze a setup where firms decide on size and location of production facilities and make production decisions that are constrained by the invested capacities, taking into account regionally differentiated prices reflecting network constraints.

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We provide general conditions that allow to establish uniqueness of the resulting market equilibrium under perfect competition, characterize this equilibrium, and provide an intuitive example. In an extension we show that our results still hold if strategic behavior of firms is approximated based on the conjectural variations approach, analogously to the approach chosen, e.g., by Wogrin et al. (2013).

To the best of our knowledge, our contribution is the first to establish uniqueness of the peak-load pricing equilibrium on a network. This is an important cornerstone to the multilevel analysis of situations where competitive firms have to make production and investment decisions facing network constraints. As it is well acknowledged in the literature, multiple solutions of lower level problems hinder the solution, interpretation, and comparison of results obtained in a multilevel context; see, e.g., Dempe (2002), Colson et al. (2007), or Gabriel et al. (2012).¹ Our result is thus important to meaningfully analyze energy policy options in computational equilibrium models, which include network expansion plans or alternative regulatory regimes.

It should be noted that our approach takes into account scarce transmission capacity of each network link for any network structure. Our current results do not cover cases where further technical constraints, that might impose additional restrictions on the feasibility of physical flows, are reflected in spot market prices. A prominent example is the consideration of a fully-fledged physical model upon the determination of spot market prices, as it is practiced in a system with nodal pricing. Since the consideration of more complex flow models is out of scope of this paper, it is a topic of future research.

Our work contributes to several strands of literature. First, it directly extends the peak-load pricing literature to peak-load pricing on a network. The seminal contributions to the analysis of peak-load pricing date back to Boiteux (1949) and Steiner (1957). For a more recent summary of the main findings and contributions see Crew et al. (1995). These contributions establish existence and uniqueness of the perfectly competitive market equilibrium in the absence of network constraints. More recently this literature has also been extended to the case of strategic firms, e.g., by Murphy and Smeers (2005), Hu and Ralph (2007), Zöttl (2010), Grimm and Zöttl (2013), or Wogrin et al. (2013). Only Zöttl (2010) and Grimm and Zöttl (2013) consider specific conditions that guarantee uniqueness of the resulting market equilibrium with strategic firms. In the general case with multiple and discrete production technologies, however, uniqueness cannot be obtained in a framework with strategic firms, not even in the absence of network restrictions. In our contribution we thus chose to approximate the case of strategic interaction by a conjectural variations approach, similar to the one applied recently by Wogrin et al. (2013), which allows to establish a unique solution.

Our article also contributes to the literature on market interaction in the presence of network constraints. This literature dates back to early contributions by Vickrey (1971) and Bohn et al. (1984), who were among the first to study optimal pricing on a network with several spatially located consumers and producers. Hogan (2012) or Chao and Peck (1996) build on those seminal contributions to analyze optimal transmission pricing in electricity markets under nodal pricing—a regime that nowadays is used in various electricity markets in the US, Canada, and some other countries. European and Australian electricity markets, however, predominantly use a system of zonal prices, where only predetermined “available transfer capacities”

¹The problem of multiplicity plagues the literature on computational multilevel analysis and is addressed in various different ways. Several recent articles try to find ways allowing to at least partially overcome the problems induced by multiplicity of lower-level solutions. Compare, e.g., Ralph and Xu (2011) for two-stage stochastic programs or Huppmann and Egerer (2015) and Ruiz and Conejo (2015), which apply a specific equilibrium selection mechanism.

between zones are taken into account upon trading at the spot market. For a discussion also see Pérez-Arriaga and Olmos (2005), Ehrenmann and Smeers (2005), or Ehrenmann and Neuhoff (2009). All those studies do not focus on uniqueness of the problem under consideration and most importantly abstract from firms' endogenous choice of production capacities, which is at the heart of our analysis.

The paper is organized as follows. In Sect. 2 we introduce the notation used throughout the paper and the considered peak-load pricing model is stated. Moreover, an equivalent reformulation of this model is given, which is used in Sect. 3 to prove the uniqueness of solutions of the peak-load pricing framework. Section 4 provides an illustrative example of our findings. Finally, Sect. 5 concludes and states some topics of further research.

2. A FRAMEWORK OF PEAK-LOAD PRICING ON A NETWORK

2.1. Notation and Model Formulation. We consider a general transport network modeled by a connected and directed graph $G = (N, A)$ with node set N and arc set A . Flow on arc a is denoted by f_a , which is limited by the arc capacity $f_a^+ \in \mathbb{R}^+$, i.e., $|f_a| \leq f_a^+$. Throughout the paper we make use of the standard δ -notation, i.e., the set of in- and outgoing arcs of a node set $M \subseteq N$ is given by

$$\begin{aligned}\delta^{\text{in}}(M) &:= \{a = (m, n) \in A : m \notin M, n \in M\}, \\ \delta^{\text{out}}(M) &:= \{a = (n, m) \in A : n \in M, m \notin M\}.\end{aligned}$$

The time horizon (or scenario set) that we consider in our peak-load pricing framework is given as an interval $T = [t_0, t_e] \subset \mathbb{R}$ with $t_0 < t_e$. Demand $d_{t,n} \geq 0$ is located at every node $n \in N$. Elastic demand at node $n \in N$ and time $t \in T$ is modeled by a continuous function $p_{t,n} : \mathbb{R}^+ \rightarrow \mathbb{R}$. For later reference we note the following additional assumption on the demand functions:

Assumption 1. *All demand functions $p_{t,n}(d)$ are strictly decreasing, i.e., $p'_{t,n}(d) < 0$.*

Under Assumption 1, we can specify the definition of our demand functions to $p_{t,n} : [0, d_{t,n}^+] \rightarrow \mathbb{R}^+$, where $d_{t,n}^+$ is the unique root of $p_{t,n}$. Further note that the gross consumer surplus, which is defined as

$$\int_0^{d_{t,n}} p_{t,n}(x) dx,$$

is concave under Assumption 1.

Moreover, at every node $n \in N$ a production technology is located that is characterized by its variable production costs $c_n^{\text{var}} \in \mathbb{R}^+$ and its capacity investment costs $c_n^{\text{inv}} \in \mathbb{R}^+$. Production at time $t \in T$ is denoted by $y_{t,n} \in \mathbb{R}^+$ and capacity by $\bar{y}_n \in \mathbb{R}^+$, i.e., capacity is constant over time. Since actual production is nonnegative and restricted by the corresponding capacity, we have $0 \leq y_{t,n} \leq \bar{y}_n$. We remark that we analyze a perfectly competitive environment, i.e., all firms are price takers.²

Again, for later reference, we formalize an additional assumption on the variable production costs.

Assumption 2. *All variable production costs $c_n^{\text{var}}, n \in N$, are pairwise distinct.*

²Under the assumption of strategic firms it is easy to show that multiple equilibria would obtain in the present setup. Typically, papers that focus on strategic interaction analyze much simpler frameworks—and often still find multiple equilibria. Since the focus of our paper is to show uniqueness of the market game (in order to develop a basis to analyze policy proposals with computational equilibrium setups), we have to restrict attention to the case of perfect competition. In order to shed lights on a world with positive markups, in Sect. 3.3 we use a simplified approach that draws on the idea of conjectural variations; see, e.g., Giocoli (2003).

We now state the optimization model that is considered throughout the paper. Basically, we want to solve the multicriteria optimization problem, in which every firm (located at node $n \in N$) maximizes its profit

$$\chi_n := \int_{t_0}^{t_e} p_{t,n}(d_{t,n}) d_{t,n} dt - \int_{t_0}^{t_e} c_n^{\text{var}} y_{t,n} dt - c_n^{\text{inv}} \bar{y}_n$$

subject to the set of common constraints

$$\sum_{a \in \delta^{\text{in}}(n)} f_{t,a} - \sum_{a \in \delta^{\text{out}}(n)} f_{t,a} - d_{t,n} + y_{t,n} = 0 \quad \text{for all } t \in T, n \in N, \quad (1a)$$

$$-f_a^+ \leq f_{t,a} \leq f_a^+ \quad \text{for all } t \in T, a \in A, \quad (1b)$$

$$0 \leq y_{t,n} \leq \bar{y}_n \quad \text{for all } t \in T, n \in N, \quad (1c)$$

$$0 \leq d_{t,n} \quad \text{for all } t \in T, n \in N. \quad (1d)$$

Constraint (1a) models flow balance for every node in every scenario. Constraint (1c) states production restrictions according to capacity investment and (1b) ensures that flow on arcs does not exceed the corresponding arc capacities. Note that capacity investment decisions are taken once for every node and are thus independent of a specific time t . Here and in what follows, quantities without node or arc indices denote the vector of the corresponding quantities; e.g., $d := (d_{t,n})_{t \in T, n \in N}$ is the vector of demands at all times $t \in T$ and all nodes $n \in N$. Thus, the considered multicriteria optimization problem reads

$$\max_{d, y, \bar{y}, f} (\chi_n(d, y, \bar{y}, f))_{n \in N} \quad \text{s.t.} \quad (1). \quad (2)$$

Note that due to the assumption of perfect competition $\partial_{d_{t,n}} p_{t,n} = 0$ holds. In this case, by the first theorem of welfare economics, the above model yields the same outcome as the corresponding welfare maximization problem:

$$\max_{d, y, \bar{y}, f} \int_{t_0}^{t_e} \sum_{n \in N} \int_0^{d_{t,n}} p_{t,n}(x) dx dt - \int_{t_0}^{t_e} \sum_{n \in N} c_n^{\text{var}} y_{t,n} dt - \sum_{n \in N} c_n^{\text{inv}} \bar{y}_n \quad (3a)$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{in}}(n)} f_{t,a} - \sum_{a \in \delta^{\text{out}}(n)} f_{t,a} - d_{t,n} + y_{t,n} = 0 \quad \text{for all } t \in T, n \in N, \quad (3b)$$

$$-f_a^+ \leq f_{t,a} \leq f_a^+ \quad \text{for all } t \in T, a \in A, \quad (3c)$$

$$0 \leq y_{t,n} \leq \bar{y}_n \quad \text{for all } t \in T, n \in N, \quad (3d)$$

$$0 \leq d_{t,n} \quad \text{for all } t \in T, n \in N. \quad (3e)$$

Note that the objective function (3a) models total social welfare, which is the difference of gross consumer surplus aggregated over all scenarios (first term) and production as well as capacity investment costs (second and third term). The equivalence can be easily shown by comparing the KKT conditions of the multicriteria optimization problem (2) with the KKT conditions of the welfare optimization problem (3). We refer the interested reader to Ehrgott (2006) for a discussion of KKT conditions of multicriteria models.

We remark that we choose to state our peak-load pricing model (3) in continuous time since it allows a more straight forward formulation of the theory presented in Sect. 3; see, e.g., Assumptions 3 and 4.

Model (3) is a concave optimization problem over a polytopal feasible set, where the boundedness follows from the production constraints (3d).

2.2. Model Reformulation. Our goal is to show that the presented peak-load pricing framework has a unique solution. To this end, we equivalently reformulate

Model (3) as

$$\max_{\bar{y}} \psi(\bar{y}) := \int_{t_0}^{t_e} \phi_t(\bar{y}) dt - \sum_{n \in N} c_n^{\text{inv}} \bar{y}_n, \quad (4)$$

where $\phi_t(\bar{y})$ is defined as the optimal value function of the subproblem for fixed time t :

$$\phi_t(\bar{y}) := \max_{d_t, y_t, f_t} \sum_{n \in N} \int_0^{d_{t,n}} p_{t,n}(x) dx - \sum_{n \in N} c_n^{\text{var}} y_{t,n} \quad (5a)$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{in}}(n)} f_{t,a} - \sum_{a \in \delta^{\text{out}}(n)} f_{t,a} - d_{t,n} + y_{t,n} = 0 \quad \text{for all } n \in N, \quad (5b)$$

$$-f_a^+ \leq f_{t,a} \leq f_a^+ \quad \text{for all } a \in A, \quad (5c)$$

$$0 \leq y_{t,n} \leq \bar{y}_n \quad \text{for all } n \in N, \quad (5d)$$

$$0 \leq d_{t,n} \quad \text{for all } n \in N. \quad (5e)$$

Note that the master problem (4) is an unconstrained optimization problem and does not explicitly depend on the network flow model. Subproblem (5) is again a concave maximization problem over a polytopal feasible set in which the capacity investments are fixed.

This reformulated model has a strong similarity to a two-stage stochastic program. If we interpret the time integral (after normalization) as the expected welfare we see that in the first stage we choose long-term capacity investments which then parameterize the second stage, in which production and demand realize in dependence on the scenarios.

3. EXISTENCE AND UNIQUENESS

Since existence of solutions is trivial (e.g., $(d, y, f) = (0, 0, 0)$ is always feasible), we focus on uniqueness of the solution. To this end, we exploit the decomposition into a master- and a subproblem introduced in Sect. 2.2. First, we prove uniqueness of the Subproblem (5) in Sect. 3.1 and then show, using this result, the uniqueness of the master problem (4) in Sect. 3.2. By this, it directly follows that the original model (3) has a unique solution.

3.1. The Subproblem. We begin our considerations about the subproblem with the repetition of the simple observation that the subproblem is a concave maximization problem over a flow polyhedron with additional restrictions on the production variables y .³ The latter implies that the feasible set is a polytope. By concavity of the objective function and convexity of the feasible set, we have the following lemma.

Lemma 1. *Suppose Assumption 1 holds. Then, exactly one of the two following cases occurs:*

- (1) *There exist demands d^* and productions y^* such that every optimal solution of Subproblem (5) is of the form (d^*, y^*, f) for some flow f .*
- (2) *Subproblem (5) has an infinite family of optimal solutions $(d^\lambda, y^\lambda, f^\lambda)$ for $\lambda \in [0, 1]$ of the form*

$$(d^\lambda, y^\lambda, f^\lambda) := \lambda(d^1, y^1, f^1) + (1 - \lambda)(d^0, y^0, f^0),$$

where (d^0, y^0, f^0) and (d^1, y^1, f^1) are solutions to Subproblem (5) with $(d^0, y^0) \neq (d^1, y^1)$.

³For the sake of simplicity, we drop the index t in this section. That is, e.g., $y = (y_{t,n})_{n \in N}$ for the currently considered $t \in T$.

The next step is to show that it is sufficient to prove that there is a unique solution if we fix the binding inequalities. For this, we define sets of active indices in dependence of a feasible point $z := (d, y, f)$:

$$\begin{aligned}\mathcal{A}_f^-(z) &:= \{a \in A: f_a = -f_a^+\}, & \mathcal{A}_f^+(z) &:= \{a \in A: f_a = f_a^+\}, \\ \mathcal{A}_y^-(z) &:= \{n \in N: y_n = 0\}, & \mathcal{A}_y^+(z) &:= \{n \in N: y_n = \bar{y}_n\}, \\ \mathcal{A}_d^-(z) &:= \{n \in N: d_n = 0\}.\end{aligned}$$

We can now state the following lemma:

Lemma 2. *Suppose Assumption 1 holds. Then, exactly one of the two following cases occurs:*

- (1) *There exist demands d^* and productions y^* such that every optimal solution of Subproblem (5) is of the form (d^*, y^*, f) for some flow f .*
- (2) *There exist two optimal solutions $z' := (d', y', f')$ and $z'' := (d'', y'', f'')$ of Subproblem (5) with $(d', y') \neq (d'', y'')$ and*

$$\begin{aligned}\mathcal{A}_y^-(z') &= \mathcal{A}_y^-(z''), & \mathcal{A}_y^+(z') &= \mathcal{A}_y^+(z''), \\ \mathcal{A}_f^-(z') &= \mathcal{A}_f^-(z''), & \mathcal{A}_f^+(z') &= \mathcal{A}_f^+(z''), \\ \mathcal{A}_d^-(z') &= \mathcal{A}_d^-(z'').\end{aligned}$$

This lemma follows directly by combining Lemma 1 and the following observation: In the interior of the solution segment from the second case of Lemma 1, the binding patterns coincide, hence we can always choose suitable solutions. This can be deduced from the following proposition.

Proposition 1. *Let $z^\lambda := (d^\lambda, y^\lambda, f^\lambda)$ be an infinite family of optimal solutions for $\lambda \in [0, 1]$ of the form*

$$(d^\lambda, y^\lambda, f^\lambda) := \lambda(d^1, y^1, f^1) + (1 - \lambda)(d^0, y^0, f^0).$$

Let $c^T z \leq r$ be a linear inequality such that $c^T z^\lambda \leq r$ holds for all $\lambda \in [0, 1]$. Then, exactly one of the following cases occurs:

- (1) *$c^T z^\lambda = r$ for all $\lambda \in [0, 1]$,*
- (2) *$c^T z^\lambda < r$ for all $\lambda \in (0, 1)$.*

Proof. By the definition of z^λ we can write $c^T z^\lambda = \lambda c^T z^1 + (1 - \lambda)c^T z^0$. This leads to the following observations: If $c^T z^0 = c^T z^1 = r$, we are in Case 1 and if $c^T z^0 < r$ and $c^T z^1 < r$ both hold, we are in Case 2. Hence, it remains to treat the case where exactly one of $c^T z^0 = r$ or $c^T z^1 = r$ holds. Without loss of generality we assume that $c^T z^0 = r$ and $c^T z^1 < r$ hold. Then, for $\lambda > 0$ we have

$$c^T z^\lambda = \lambda c^T z^1 + (1 - \lambda)c^T z^0 = \lambda c^T z^1 + (1 - \lambda)r < \lambda r + (1 - \lambda)r = r.$$

Thus, we are in Case 2. □

For the following it is advantageous to use the concept of price clusters.

Definition 1. Given a solution z of Subproblem (5), we say that a partition $\mathcal{C} = \{C_i\}_{i=1}^I$ partitions the node set N into *price clusters*, if for all $C \in \mathcal{C}$ holds, that for all nodes in the cluster C the shadow prices of the flow conservation constraints (i.e., the dual variables of Constraints (5b)) are equal. We also write $\mathcal{C}(z)$ to emphasize the dependence on the solution z . An arc $a = (n, m)$ is called an *inter-cluster arc*, if $n \in C_i$ and $m \in C_j$ with $i \neq j$ and we denote the set of inter-cluster arcs by A^{inter} .

We now want to use a result shown by Schewe and Schmidt (2015) in a slightly different situation; namely that price clusters of the network are characterized by the binding constraints in (5c). For this we introduce another partition.

Definition 2. Given a solution z of Subproblem (5), we say that the partition $\mathcal{C} = \{C_i\}_{i=1}^I$ of the node set N is the *flow-induced partition*, if each C_i is a connected component of the graph $\tilde{G}(z) = (V, A \setminus A^{\text{sat}})$, where $A^{\text{sat}} := \{a \in A : |f_a| = f_a^+\}$.

With this definition, the required result reads as follows:

Theorem 1. *Let $z^* := (d^*, y^*, f^*)$ be an optimal solution of Subproblem (5) and let $\mathcal{C}(z^*)$ be the corresponding flow-induced partition. Then,*

$$\phi(\bar{y}) = \max_{d, y} \sum_{n \in N} \int_0^{d_n} p_n(x) dx - \sum_{n \in N} c_n^{\text{var}} y_n \quad (6a)$$

$$\text{s.t.} \quad \sum_{n \in C} d_n - \sum_{n \in C} y_n = \hat{f}_C \quad \text{for all } C \in \mathcal{C}(z^*), \quad (6b)$$

$$0 \leq y_n \leq \bar{y}_n \quad \text{for all } n \in N, \quad (6c)$$

$$d_n \geq 0 \quad \text{for all } n \in N, \quad (6d)$$

where $\hat{f}_C = \sum_{a \in \delta^{\text{in}}(C)} f_a^+ - \sum_{a \in \delta^{\text{out}}(C)} f_a^+$ is the total in- or outflow of zone C . This implies that $\mathcal{C}(z^*)$ is a partition into price clusters.

Proof. The proof is given in Appendix A. \square

Thus, Lemma 2 combined with the cited result states that whenever there exist two different optimal solutions, there also exist two different solutions with the same price clusters. Moreover, the flows between these clusters are unique since they are at their bounds.

Lemma 3. *Suppose Assumption 1 holds. Then, exactly one of the two following cases occurs:*

- (1) *There exist demands d^* and productions y^* such that every optimal solution of Subproblem (5) is of the form (d^*, y^*, f) for some flow f .*
- (2) *There exist two optimal solutions $z' := (d', y', f')$ and $z'' := (d'', y'', f'')$ of Subproblem (5) with $(d', y') \neq (d'', y'')$ such that*
 - (a) $\mathcal{C}(z') = \mathcal{C}(z'')$ and
 - (b) *for z' and z'' it holds that Constraint (5c) is tight for an arc a if and only if a is an inter-cluster arc.*

Proof. The lemma follows directly from Lemma 2 with the following additional argument: Assume there exists an arc $a = (n, m)$ with $a \in \mathcal{A}_f^+(z')$ and a is not an inter-cluster arc, that is $n, m \in C$ for some $C \in \mathcal{C}$. We show that we can modify solution z' so that we obtain an optimal solution \tilde{z}' with the same activity pattern with the exception that $\mathcal{A}_{\tilde{f}'}^+(z') = \mathcal{A}_f^+(z') \setminus \{a\}$. As a is not an inter-cluster arc, there must exist a path P connecting n and m completely lying in cluster C such that for all $a \in P$ it holds that $a \notin \mathcal{A}_f^+(z') \cup \mathcal{A}_f^-(z')$, i.e., no flow bound on P is active. That means it must be possible to send an additional amount of flow ε along P without violating any bounds. Hence, we can reduce the amount of flow sent along a by $\varepsilon/2$ and send the same amount along path P . This gives us a new flow \tilde{f}' . Set $\tilde{z}' := (d', y', \tilde{f}')$, then the flow bound for arc a is no longer active. As a was an arbitrary non-inter-cluster arc, we can iterate this procedure until only flow bounds on inter-cluster arcs are attained. This can be done with both z' and z'' and thus we obtain the desired result. \square

The last lemma implies that the ambiguity of solutions have to be “inside” the price clusters. Thus, we only have to consider these clusters in the following. Since

the network constraints do not play a role within the price clusters, Subproblem (5) for a single cluster reduces to the concave maximization problem

$$\max_{d,y} \quad \sum_{n \in C} \int_0^{d_n} p_n(x) dx - \sum_{n \in C} c_n^{\text{var}} y_n \quad (7a)$$

$$\text{s.t.} \quad \sum_{n \in C} d_n - \sum_{n \in C} y_n = \hat{f}_C, \quad (7b)$$

$$0 \leq y_n \leq \bar{y}_n \quad \text{for all } n \in C, \quad (7c)$$

$$d_n \geq 0 \quad \text{for all } n \in C, \quad (7d)$$

where $C \subseteq N$ is the set of nodes of the considered price cluster and \hat{f}_C is total in- or outflow of this cluster; see Theorem 1. The KKT conditions of this problem comprise the dual feasibility conditions

$$\begin{aligned} p_n(d_n) + \alpha + \gamma_n &= 0 \quad \text{for all } n \in C, \\ -c_n^{\text{var}} - \alpha + \beta_n^- - \beta_n^+ &= 0 \quad \text{for all } n \in C, \end{aligned}$$

where $\alpha \in \mathbb{R}$ is the dual variable of Constraint (7b), $\beta_n^-, \beta_n^+, n \in N$, are the dual variables of the lower and upper production bounds in (7c), and γ_n is the dual variable of the demand bounds (7d). This immediately implies a single price $p_C := -\alpha$ with $p_C = p_n(d_n)$ for all $n \in C$ with $d_n > 0$. Nodes n with $d_n = 0$ do not contribute to the objective value and hence their price can be ignored. Moreover,

$$p_C - c_n^{\text{var}} + \beta_n^- - \beta_n^+ = 0 \quad (8)$$

holds for all $n \in C$ with $d_n > 0$.

Our goal is now to show that productions and demands inside a cluster are uniquely determined. The flow values within the price clusters, however, are not unique, since we can always modify a solution with a flow along a cycle as long as we stay inside the bounds. However, these ambiguous flows do not interfere with the optimal demand and production values and thus do not influence the objective function value. We summarize our findings in the following theorem:

Theorem 2. *Suppose Assumptions 1 and 2 hold. Then, there are unique demands d_C^* and production y_C^* such that every optimal solution of Model (7) has the form (d_C^*, y_C^*, f_C) for some f_C .*

Proof. Assume that the price inside the price cluster is given by p_C . As the demand functions p_n are strictly decreasing and thus bijective, there is a unique demand d_n for every $n \in C$. Hence, there exists a function $d_C(p)$ that maps every price p to the unique aggregate demand at that price point. We define $\hat{d}_C(p) := d_C(p) - \hat{f}_C$. As the demand function for each node is strictly decreasing, the aggregated function $\hat{d}_C(p)$ is strictly decreasing as well.

On the production side we can see that given a p_C we can immediately determine (by using Condition (8)) which nodes $n \in C$ are definitely not producing ($c_n^{\text{var}} > p_C$), the ones definitely producing at maximum capacity ($c_n^{\text{var}} < p_C$), and the ones where the production amount is indeterminate, that is between 0 and \bar{y}_n ($c_n^{\text{var}} = p_C$). Under Assumption 2 there exists at most one node such that $c_n^{\text{var}} = p_C$. Hence for all nodes except at most one, the price p_C uniquely determines the production values of the nodes. Moreover, we obtain two functions $y_C^{\text{min}}(p)$ and $y_C^{\text{max}}(p)$ which are the minimal, resp. maximal, production in the price cluster at a given price p . Both of these functions are monotonically increasing. If we intersect the functions \hat{d}_C and y_C^{min} , we observe that they have at most one intersection point and analogously for the functions \hat{d}_C and y_C^{max} . From the construction of y_C^{min} and y_C^{max} it then follows that there is exactly one price p_C^* such that $y_C^{\text{min}} \leq \hat{d}_C(p_C^*) \leq y_C^{\text{max}}$. Hence,

every optimal solution of our problem yields the same price p_C^* . From the discussion of the first paragraph the uniqueness of the demands then follows directly. For the production the uniqueness is also clear for all nodes except at most one. The production of this last node, however, is also uniquely determined by the market clearing constraint. \square

The proof allows us also to conclude that the dual variables β are unique as well; see Condition (8).

Corollary 1. *Suppose Assumption 1 and 2 hold. Then, the values of the dual variables β_n^\pm are unique for all nodes $n \in N$.*

All in all, we have the following result concerning Subproblem (5):

Theorem 3. *Suppose Assumption 1 and 2 hold. Furthermore, let $\mathcal{C} = \{C_i\}_{i=1}^I$ be the unique partition of the node set into price clusters, let $A^{\text{inter}} := \{a = (n, m) \in A : n \in C_i, m \in C_j, i \neq j\}$ be the set of inter-cluster arcs. Then, the solution (d, y, f) of Subproblem (5) is unique in $(d, y, f_{A^{\text{inter}}})$.*

Proof. By Lemma 3 we need to consider two cases. In the first case we are done. We need to show that the second case cannot occur. This, however, follows directly from Theorem 2. \square

3.2. The Master Problem. In this section we prove that—given the results of the preceding section—the master problem (4) has a unique solution. To this end, we prove that the Hessian $H(\bar{y})$ of ψ is negative definite. Since the linear terms $\sum_{n \in N} c_n^{\text{inv}} \bar{y}_n$ in (4) vanish in second order, the Hessian of ψ is completely given by the Hessian of the integral terms. Thus, we have to compute the second derivative $H(\bar{y})$ w.r.t. \bar{y} of

$$\int_{t_0}^{t_e} \phi_t(\bar{y}) dt. \quad (9)$$

We split this section into two parts: In Sect. 3.2.1, we determine the second derivative w.r.t. \bar{y} of $\phi_t(\bar{y})$ for a fixed time t . The subsequent Sect. 3.2.2 then considers the second derivative of (9).

3.2.1. The Single-Scenario Case. In this section we compute the Hessian for a fixed time t , i.e., the Hessian

$$H_t(\bar{y}) = \nabla_{\bar{y}\bar{y}}^2 \phi_t(\bar{y})$$

of $\phi_t(\bar{y})$. The first-order partial derivatives are known from standard sensitivity analysis (see, e.g., Boyd and Vandenberghe (2004)) of convex optimization:

$$\frac{\partial}{\partial \bar{y}_n} \phi_t(\bar{y}) = \beta_{t,n}^+ \quad \text{for all } n \in N,$$

where $\beta_{t,n}^+$ is the dual variable corresponding to the upper bound in Constraint (5d). Thus, we now have to compute the derivative of $\beta_{t,n}^+$ with respect to \bar{y}_m for all $n, m \in N$. In the following we require a series of partitions of the node set and the time horizon. An overview over all partitions and subsets is given in Table 1. For a fixed time t , we obtain a partition $\mathcal{C}_t(\bar{y}) = \{C_{t,i}(\bar{y})\}_{i=1}^{I_t}$ of the node set N into price clusters as described in the last section. Now, we consider a single price cluster $C_{t,i}(\bar{y})$, i.e., we fix some $i \in \{1, \dots, I_t\}$ for the moment. It can be easily verified that the first-order conditions of Subproblem (5) imply

$$\beta_{t,n}^+ = \begin{cases} p_{t,i} - c_n^{\text{var}}, & \text{if } y_{t,n} = \bar{y}_n, \\ 0, & \text{if } y_{t,n} < \bar{y}_n, \end{cases}$$

where $p_{t,i}$ is the price of cluster $C_{t,i}(\bar{y})$. The derivative of $\beta_{t,n}^+$ w.r.t. \bar{y}_m is obviously zero for every node $m \in N$ in the second case. The first case, i.e., the case in which

TABLE 1. Subsets of the node set and time horizon as well as (blocks of) considered Hessian matrices (Remark: All these sets depend on \bar{y})

Set	Explanation
$C_{t,i} \subseteq N$	i th price cluster at time t
$\mathcal{C}_t = \{C_{t,i}\}_{i=1}^{I_t}$	Partition of the node set into price clusters for time t
$A_{t,i} \subseteq C_{t,i}$	Nodes of price cluster $i \in \{1, \dots, I_\tau\}$ in time t with $\beta_{t,n} > 0$
\hat{T}	Times t where solutions of Problem (10) do not satisfy strict complementarity
$T_\tau \subseteq T$	Times t with equal price clusters $C_{t,i}$
$\mathcal{T} = \{T_\tau\}_\tau$	Price cluster specific time horizon partition
$T_{\tau,j} \subseteq T_\tau$	times t with equal price clusters and equal binding production nodes
$\{T_{\tau,j}\}_j$	Price cluster and active production nodes specific time horizon subset partition
H	Hessian of ψ
H_t	Hessian of ψ for a single time t
H_τ	Hessians of ψ for the time $t \in T_\tau$
$H_{\tau,i}$	Submatrix (block) of H_τ induced by price cluster i
$H_{\tau,i,j}$	Submatrix (block) of $H_{\tau,i}$ induced by active production nodes

$y_{t,n} = \bar{y}_n$ with $n \in C_{t,i}(\bar{y})$ holds, is more complicated. Let $A_{t,i}(\bar{y}) \subseteq C_{t,i}(\bar{y}) \subseteq N$ be the set of nodes of the price cluster $C_{t,i}(\bar{y})$ that are strictly active, i.e., all nodes $m \in N$ with $\beta_{t,m}^+ > 0$, which implies $y_{t,m} = \bar{y}_m$. As an auxiliary result we first need to compute the derivative of the total demand of a single cluster with respect to the capacity of a single node of that cluster. To this end, we first rewrite Model (7) for cluster $C_{t,i}(\bar{y})$ using the aggregated demand function $P_{t,i}$ and the total demand $D_{t,i}$.

$$\phi_{C_{t,i}}(\bar{y}) := \max_{D_{t,i}, y_{t,i}} \int_0^{D_{t,i}} P_{t,i}(x) dx - \sum_{n \in C_{t,i}} c_n^{\text{var}} y_{t,n} \quad (10a)$$

$$\text{s.t. } D_{t,i} - \sum_{n \in C_{t,i}} y_{t,n} = \hat{f}_{t,i}, \quad (10b)$$

$$0 \leq y_{t,n} \leq \bar{y}_n \quad \text{for all } n \in C_{t,i}, \quad (10c)$$

$$D_{t,i} \geq 0. \quad (10d)$$

Proposition 2. *Let $(D, y; \alpha, \beta^\pm, \gamma^-)$ be an optimal solution of Problem (10) such that strict complementarity holds. Let $n^* \in C_{t,i}(\bar{y})$ be the node with largest variable costs in cluster $C_{t,i}(\bar{y})$ with $y_{n^*} > 0$. If $\gamma^- > 0$ or $\beta_{n^*}^+ = 0$ then*

$$\frac{\partial D}{\partial \bar{y}_n} = 0 \quad \text{for all } n \in C_{t,i}(\bar{y}).$$

If, however, $\gamma^- = 0$ and $\beta_{n^}^+ > 0$ holds, then for all $n \in C_{t,i}(\bar{y})$, we have*

$$\frac{\partial D}{\partial \bar{y}_n} = \begin{cases} 1, & \text{if } y_n > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. After elimination of the dual variables of Constraint (10b), the KKT conditions of Problem (10) contain the following equations:

$$\begin{aligned} P(D) - c_n^{\text{var}} + \beta_n^- - \beta_n^+ + \gamma^- &= 0 \quad \text{for all } n \in C_{t,i}(\bar{y}), \\ D - \sum_{n \in C} y_n - \hat{f}_{t,i} &= 0, \\ \beta_n^- y_n &= 0 \quad \text{for all } n \in C_{t,i}(\bar{y}), \\ \beta_n^+ (\bar{y}_n - y_n) &= 0 \quad \text{for all } n \in C_{t,i}(\bar{y}), \\ \gamma^- D &= 0. \end{aligned}$$

This is a system $F(x; \bar{y}_n) = 0$ of equations with $x = (D, y, \beta^\pm, \gamma^-)$. Since strict complementarity holds we may apply the implicit function theorem, yielding

$$J_x F \cdot J_{\bar{y}_n} x = -J_{\bar{y}_n} F,$$

where, e.g., $J_x F$ denotes the Jacobian of F with respect to x . Solving this system of equations yields the claim. \square

We observe that

$$\phi_t(\bar{y}) = \sum_{i=1}^{I_t} \phi_{t,i}(\bar{y}). \quad (11)$$

holds. Now we are able to compute the second partial derivatives of $\phi_t(\bar{y})$.

Lemma 4. *Let \bar{y} and t be given and assume that the solutions of Problem (10) fulfill strict complementarity for all $i \in \{1, \dots, I_t\}$. If $n \in C_{t,i}(\bar{y})$ and $m \in C_{t,j}(\bar{y})$ with $i \neq j$, then*

$$\frac{\partial}{\partial \bar{y}_m} \frac{\partial}{\partial \bar{y}_n} \phi_t(\bar{y}) = 0. \quad (12)$$

If $n, m \in C_{t,i}(\bar{y})$ and $\gamma^- > 0$ or $\beta_{n^}^+ = 0$, where $\gamma^-, \beta_{n^*}^+$ are the respective dual variables of Problem (10) for cluster $C_{t,i}(\bar{y})$ and n^* is defined as in Proposition 2, then*

$$\frac{\partial}{\partial \bar{y}_m} \frac{\partial}{\partial \bar{y}_n} \phi_t(\bar{y}) = 0. \quad (13)$$

Otherwise, i.e., $\gamma^- = 0$ and $\beta_{n^}^+ > 0$, we have*

$$\frac{\partial}{\partial \bar{y}_m} \frac{\partial}{\partial \bar{y}_n} \phi_t(\bar{y}) = \begin{cases} B_{t,i}(\bar{y}), & \text{if } n, m \in A_{t,i}(\bar{y}), \\ 0, & \text{otherwise,} \end{cases} \quad (14)$$

where $B_{t,i}(\bar{y}) < 0$ is the negative slope of the aggregated demand function $P_{t,i}(\bar{y})$ at the total demand $D_{t,i}(\bar{y})$ of price cluster $C_{t,i}(\bar{y})$.

Proof. Equation (12) follows directly from Equation (11). For the remaining cases we make the following observation:

$$\frac{\partial}{\partial \bar{y}_m} \frac{\partial}{\partial \bar{y}_n} \phi_t(\bar{y}) = \frac{\partial}{\partial \bar{y}_m} \beta_{t,n}^+.$$

The KKT conditions of Problem (10) imply

$$\beta_{t,n}^+ = P_{t,i}(D_{t,i}(\bar{y})) - c_n^{\text{var}} \quad \text{for all } n \in C_{t,i}(\bar{y}) \text{ with } y_{t,n} > 0.$$

Thus, for $n \in C_{t,i}$ with $y_{t,n} > 0$ we can write

$$\frac{\partial}{\partial \bar{y}_m} \beta_{t,n}^+ = \frac{\partial}{\partial \bar{y}_m} P_{t,i}(D_{t,i}(\bar{y})) = \frac{\partial}{\partial D_{t,i}} P_{t,i}(D_{t,i}(\bar{y})) \frac{\partial}{\partial \bar{y}_m} D_{t,i}(\bar{y}) = B_{t,i}(\bar{y}) \frac{\partial}{\partial \bar{y}_m} D_{t,i}(\bar{y}),$$

where $D_{t,i}(\bar{y})$ is the (unique) total demand in an optimal solution of Model (10) for price cluster $C_{t,i}(\bar{y})$ in dependence on \bar{y} .

The remaining Equations (13) and (14) follow directly from Proposition 2. \square

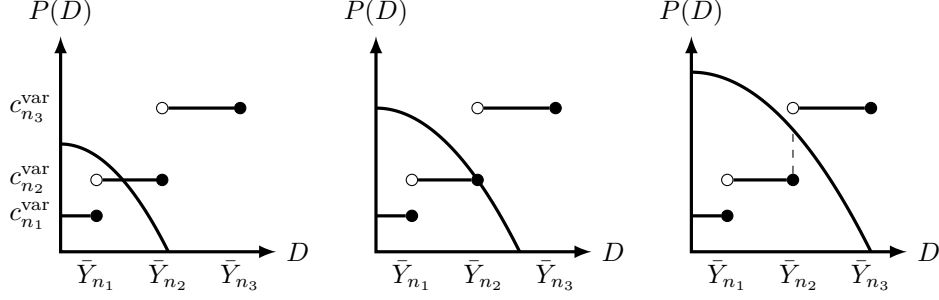


FIGURE 1. Illustration of Proposition 2 and Assumption 3; $\bar{Y}_i := \sum_{k=1}^i \bar{y}_k$. Strict complementarity holds in the left and right figure, whereas it is violated in the middle case.

We write down the necessary property from the preceding lemma.

Assumption 3. For \bar{y} let $\hat{T}(\bar{y})$ be set of all $t \in T$ such that there exists a price cluster $i \in \{1, \dots, I_t\}$ where the unique solution of Problem (10) does not satisfy strict complementarity. We assume that $\hat{T}(\bar{y})$ has measure zero for all \bar{y} .

Before we turn to the multi-scenario case, we briefly discuss the mathematical necessity of Assumption 3 and illustrate the economic interpretation of Proposition 2 and strict complementarity (or its violation) using the example of the production constraints $y_n \leq \bar{y}_n$ and their dual variables $\beta_n^+ \geq 0$. We again drop the time index for better readability. Figure 1 illustrates three possible aggregated demand functions (continuous and strictly decreasing curves) and a single aggregated supply function for a price cluster. Total demand is positive in all three cases. The price cluster equilibrium in the first case (left figure) is characterized by the intersection of the aggregated demand curve and the variable production costs of the second cheapest producer, say n_2 . In this case the production of n_2 fulfills $y_{n_2} \in (0, \bar{y}_{n_2})$, i.e., $\beta_n^- = \beta_n^+ = 0$, and strict complementarity holds. Dual feasibility then yields $P(D) = c_{n_2}^{\text{var}}$, which can also be seen in the left figure. Moreover, it can be seen that $\partial_{\bar{y}_n} D = 0$ for all nodes n . The other case satisfying strict complementarity is illustrated in the right figure: For all producing nodes m holds that $y_m = \bar{y}_m$. Moreover, $\beta_{n_2}^+ = P(D) - c_{n_2}^{\text{var}} > 0$ (dashed line) is the earning of node n_2 . The right figure also illustrates that $\partial_{\bar{y}_{n_k}} D = 1$ for all $k \leq 2$ and $\partial_{\bar{y}_{n_k}} D = 0$ for all $k > 2$ holds; see Proposition 2. The only problematic case is shown in the middle figure: Aggregated demand intersects aggregated supply at the rightmost point (\bar{Y}_{n_2}) of producer n_2 thus yielding $y_{n_2} = \bar{y}_{n_2}$ and $\beta_{n_2}^+ = 0$, i.e., strict complementarity does not hold. The mathematical severity of this case is that $\partial_{\bar{y}_{n_k}} D$ does not exist; only directional derivatives exist and equal $c_{n_3}^{\text{var}} - c_{n_2}^{\text{var}} > 0$ and 0, respectively. Finally, the middle figure suggests that this is a rare event because it only appears if the aggregated demand curve intersects the supply curve in a finite number of special points, i.e., $\bar{Y}_{n_i}, i = 1, 2, \dots$, out of a continuum of points.

3.2.2. The Multi-Scenario Case. Up to this point, we have computed the second derivative for a fixed time t . We now show that the complete Hessian

$$H(\bar{y}) = \int_{t_0}^{t_e} H_t(\bar{y}) dt$$

of (4) is negative definite. To this end, we partition the time horizon T in

$$\mathcal{T}(\bar{y}) = \{T_\tau(\bar{y})\}_\tau \cup \hat{T}(\bar{y})$$

such that for all τ all times $t \in T_\tau(\bar{y})$ have the same price clusters $\mathcal{C}_t(\bar{y})$. We remark that there only exist finitely many τ since there also exist only finitely many price cluster configurations. This allows us to state the following proposition:

Proposition 3. *Suppose Assumption 3 holds. Then, the Hessian $H(\bar{y})$ can be written as*

$$H(\bar{y}) = \int_{t_0}^{t_e} H_t(\bar{y}) dt = \sum_{\tau} \int_{T_\tau(\bar{y})} H_t(\bar{y}) dt.$$

Note that the definition of $\mathcal{T}(\bar{y})$ permits the notations $H_\tau(\bar{y})$ and $\mathcal{C}_\tau(\bar{y})$. The following proposition readily follows from (12) and states that an entry of Hessian $H_t(\bar{y})$ corresponding to two nodes n, m is zero for all nodes in different price clusters and $t \in T_\tau(\bar{y})$.

Proposition 4. *Let $(H_t(\bar{y}))_{n,m}$ denote the entry in row n and column m of the matrix $H_t(\bar{y})$. Then for all $n, m \in N$ and for all τ we have that*

$$(H_t(\bar{y}))_{n,m} = 0 \quad \text{for all } t \in T_\tau(\bar{y})$$

if $n \in C_{\tau,i}(\bar{y})$ and $m \in C_{\tau,j}(\bar{y})$ with $i \neq j$.

Note that this proposition yields a block structure of $H_t(\bar{y}), t \in T_\tau(\bar{y})$, induced by the price clusters $\mathcal{C}_\tau(\bar{y}) = \{C_{\tau,i}(\bar{y})\}_{i=1}^I$ at these times. The corresponding matrix block is denoted by $H_{\tau,i}(\bar{y})$ and, after re-ordering of the nodes, we obtain

$$H_t(\bar{y}) = \text{diag}(H_{\tau,i}(\bar{y}))_{i=1}^I.$$

We now partition the times $T_\tau(\bar{y})$ further into $\{T_{\tau,j}(\bar{y})\}_j$ such that

$$A_{i,j}(\bar{y}) := A_{t,i}(\bar{y}) = A_{t',i}(\bar{y})$$

holds for all $t, t' \in T_{\tau,j}(\bar{y})$. The following proposition is a direct consequence of these partitions.

Proposition 5. *For all $t, t' \in T_{\tau,j}(\bar{y})$ it holds that*

$$H_t(\bar{y})|_{C_{\tau,i}} = H_{t'}(\bar{y})|_{C_{\tau,i}},$$

where $H_t(\bar{y})|_{C_{\tau,i}}$ denotes the restriction of $H_t(\bar{y})$ to the block corresponding to $C_{\tau,i}$.

This proposition allows us to introduce the notation

$$H_{\tau,i,j}(\bar{y}) := H_t(\bar{y})|_{C_{\tau,i}}$$

for all $t \in T_{\tau,j}(\bar{y})$. Moreover, note that $H_{\tau,i,j}(\bar{y})$ is a matrix with a left-upper block with values $B_{\tau,i,j}(\bar{y}) < 0$ of size $|A_{i,j}(\bar{y})|$ and zeros elsewhere.

The rest of the proof is split up into two parts. First, we show that all Hessians $H_{\tau,i}(\bar{y})$ are negative semi-definite. Second, we show that under additional assumptions, there exist some $H_{\tau,i}(\bar{y})$ that are negative definite. Both results together finally imply the negative definiteness of the overall Hessian for all \bar{y} and thus the peak-load pricing model (3) has a unique solution.

Proposition 6. *For all τ and all i , the corresponding block $H_{\tau,i}(\bar{y})$ is negative semi-definite.*

Proof. Let τ and i be given. Then, by Proposition 5

$$H_{\tau,i}(\bar{y}) = \sum_j \int_{T_{\tau,j}(\bar{y})} H_t(\bar{y})|_{C_{\tau,i}} dt = \sum_j \mu(T_{\tau,j}(\bar{y})) H_{\tau,i,j}(\bar{y})$$

holds with $H_{\tau,i,j}(\bar{y})$ being rank-1-matrices in which all non-vanishing entries equal $B_{\tau,i,j}(\bar{y}) < 0$. Here, $\mu(T_{\tau,j}(\bar{y}))$ is the Lebesgue measure of $T_{\tau,j}(\bar{y})$ in T . Since $H_{\tau,i}(\bar{y})$ is now shown to be a sum of negative semi-definite matrices, this shows that $H_{\tau,i}(\bar{y})$ itself is negative semi-definite. \square

Note that from the latter proposition directly follows that $H_\tau(\bar{y})$ is negative semi-definite for all τ , since $H_\tau(\bar{y})$ is a block-diagonal matrix with blocks $H_{\tau,i}(\bar{y})$

Proposition 7. *Let τ and i be given. If the partition $\{T_{\tau,j}(\bar{y})\}_{j=1}^J$, $J = |C_{\tau,i}(\bar{y})|$, of $T_\tau(\bar{y})$ can be chosen so that*

$$A_{i,j+1}(\bar{y}) = A_{i,j}(\bar{y}) \cup \{n_{j+1}\}, \quad A_{i,1}(\bar{y}) = \{n_1\}$$

holds, where the nodes n_1, \dots, n_J are ordered in such a way that $c_{n_k}^{\text{var}} < c_{n_\ell}^{\text{var}}$ if and only if $k < \ell$ for all $1 \leq k, \ell \leq J$, and if $\mu(T_{\tau,j}(\bar{y})) > 0$ holds for all j , then $H_{\tau,i}(\bar{y})$ is negative definite.

Proof. The partition of the set of times and nodes readily implies

$$H_{\tau,i}(\bar{y}) = \sum_j \mu(T_{\tau,j}(\bar{y})) H_{\tau,i,j}(\bar{y}) =: \sum_j \tilde{H}_{\tau,i,j}(\bar{y}).$$

We now define $\tilde{B}_{\tau,i,j} := \mu(T_{\tau,j}(\bar{y})) B_{\tau,i,j}(\bar{y})$. With this notation the following holds:

$$(\tilde{H}_{\tau,i,j}(\bar{y}))_{\nu,\xi} = \begin{cases} \tilde{B}_{\tau,i,j}, & \text{if } \nu, \xi \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

We now apply Gaussian elimination: In the k th step we subtract row $J - k + 1$ from all rows 1 to $J - k + 2$. After $J - 1$ steps this yields the matrix

$$\begin{bmatrix} \tilde{B}_{\tau,i,1} & & & & & & \\ \tilde{B}_{\tau,i,2} & \tilde{B}_{\tau,i,2} & & & & & \\ \vdots & & \ddots & & & & \\ \tilde{B}_{\tau,i,J-1} & \cdots & & \tilde{B}_{\tau,i,J-1} & & & \\ \tilde{B}_{\tau,i,J} & \cdots & & & & \tilde{B}_{\tau,i,J} & \end{bmatrix}.$$

Since all diagonal elements $\tilde{B}_{\tau,i,j}$ of the resulting matrix are strictly negative the matrix is negative definite. \square

The last proposition leads us to the following assumption:

Assumption 4. *There exists a time partition index τ such that for all $i = 1, \dots, I_t$, $T_\tau(\bar{y})$ can be partitioned as $\{T_{\tau,j}(\bar{y})\}_{j=1}^J$, $J = |C_{\tau,i}(\bar{y})|$, with*

$$A_{i,j+1}(\bar{y}) = A_{i,j}(\bar{y}) \cup \{n_{j+1}\}, \quad A_{i,1}(\bar{y}) = \{n_1\},$$

where the nodes n_1, \dots, n_J are ordered in such a way that $c_{n_k}^{\text{var}} < c_{n_\ell}^{\text{var}}$ if and only if $k < \ell$ for all $1 \leq k, \ell \leq J$ and $\mu(T_{\tau,j}(\bar{y})) > 0$ holds for all j .

This assumption can be seen as a natural extension of Assumption 2. If our scenario set does not fulfill the assumption, i.e., informally speaking, that given two different nodes the following situation occurs: In almost all scenarios where they are part of the same price cluster they are always both producing at full capacity or both do not produce at all. In other words, the scenario set is not large enough to distinguish between these two nodes. Then, it is clear that the solution may not be unique. With realistic data, however, this should not occur as producers are sufficiently different and scenario sets are sufficiently large to ensure this condition.

Summing up all results of the last sections, we obtain the following main theorem:

Theorem 4. *Suppose Assumptions 1–4 hold. Then, the matrix $H(\bar{y})$ is negative definite and, thus, Model (3) has a unique solution in $(d, y, f_{A^{\text{inter}}})$.*

3.3. The Case of Market Power. As we have argued earlier, it is impossible to meaningfully analyze the proposed framework using a rigorous game theoretic approach to strategic interaction among firms. Various papers have shown that multiple equilibria already arise in a setup with strategic interaction in the absence of networks; see, e.g., Zöttl (2010). In our contribution we thus chose to approximate the case of strategic interaction by a conjectural variations approach, similar to the one applied recently by Wogrin et al. (2013), which allows to establish a unique solution. While this approach and its outcome cannot be related to a proper game structure, it nevertheless might be suitable to capture important aspects of an environment where firms manage to charge significant markups.

To this end, we replace objective function (3a)

$$\psi^1 := \int_{t_0}^{t_e} \sum_{n \in N} \int_0^{d_{t,n}} p_{t,n}(x) dx dt - \int_{t_0}^{t_e} \sum_{n \in N} c_n^{\text{var}} y_{t,n} dt - \sum_{n \in N} c_n^{\text{inv}} \bar{y}_n$$

by

$$\psi^\lambda := \lambda \psi^1 + (1 - \lambda) \psi^0, \quad \lambda \in [0, 1], \quad (15)$$

where

$$\psi^0 := \int_{t_0}^{t_e} \sum_{n \in N} p_{t,n}(d_{t,n}) d_{t,n} dt - \int_{t_0}^{t_e} \sum_{n \in N} c_n^{\text{var}} y_{t,n} dt - \sum_{n \in N} c_n^{\text{inv}} \bar{y}_n.$$

Note that this extension is a convex combination of the situation, in which competitive firms trade on a market (ψ^1) and the case of a monopoly (ψ^0). It is easily seen that this extension only affects the demand terms, i.e.,

$$\begin{aligned} \psi^\lambda &= \lambda \int_{t_0}^{t_e} \sum_{n \in N} \int_0^{d_{t,n}} p_{t,n}(x) dx dt + (1 - \lambda) \int_{t_0}^{t_e} \sum_{n \in N} p_{t,n}(d_{t,n}) d_{t,n} dt \\ &\quad - \int_{t_0}^{t_e} \sum_{n \in N} c_n^{\text{var}} y_{t,n} dt - \sum_{n \in N} c_n^{\text{inv}} \bar{y}_n \end{aligned}$$

holds. In the following, we show that all results presented so far are also valid for the case of Objective (15) under the following additional assumption:

Assumption 5. *All demand functions $p_{t,n}(d)$ fulfill Assumption 1 and the additional condition $p'_{t,n}(d) + p''_{t,n}(d)d < 0$.*

We note that in the common case where $p_{t,n}$ is a linear function, Assumption 1 directly implies Assumption 5.

Lemma 5. *It holds that*

$$\psi^\lambda = \int_{t_0}^{t_e} \sum_{n \in N} \int_0^{d_{t,n}} p_{t,n}^\lambda(x) dx dt - \int_{t_0}^{t_e} \sum_{n \in N} c_n^{\text{var}} y_{t,n} dt - \sum_{n \in N} c_n^{\text{inv}} \bar{y}_n,$$

where

$$p_{t,n}^\lambda(x) := p_{t,n}(x) + (1 - \lambda) p'_{t,n}(x)x.$$

If $p_{t,n}$ fulfills Assumption 5, then $p_{t,n}^\lambda$ fulfills Assumption 1.

Proof. We only have to consider the demand terms for fixed time t and node n separately. Then, the proof is a straight forward application of integration by parts:

$$\begin{aligned}
\int_0^{d_{t,n}} p_{t,n}^\lambda(x) dx &= \int_0^{d_{t,n}} p_{t,n}(x) + (1-\lambda)p'_{t,n}(x)x dx \\
&= \int_0^{d_{t,n}} p_{t,n}(x) dx + (1-\lambda) \int_0^{d_{t,n}} p'_{t,n}(x)x dx \\
&= \int_0^{d_{t,n}} p_{t,n}(x) dx + (1-\lambda) \left([p_{t,n}(x)x]_0^{d_{t,n}} - \int_0^{d_{t,n}} p_{t,n}(x) dx \right) \\
&= \lambda \int_0^{d_{t,n}} p_{t,n}(x) dx + (1-\lambda)p_{t,n}(d_{t,n})d_{t,n}.
\end{aligned}$$

The second claim is immediately clear. \square

This lemma shows that the model using the modified objective (15) is simply the basic peak-load pricing model (3) with demand functions $p_{t,n}$ replaced by $p_{t,n}^\lambda$, which are again strictly decreasing. Thus, all results from Section 3 also apply to the model using Objective (15).

3.4. Characterization and Discussion. We now discuss how the optimal solution of Problem (3) can be characterized. If we analyze the situation of a single scenario, we observe that prices in neighboring clusters differ by the shadow price of their saturated connecting arcs. Assume we are given two clusters C_P and C_C , where C_P supplies more than it demands and C_C demands more than it supplies. Then, the first order conditions of Problem (5) directly imply that on all arcs connecting C_P with C_C the flow direction is from C_P to C_C and that $p_{C_P} < p_{C_C}$. Thus, flow goes from the lower to the higher price.

Focusing on the full problem, we are interested in how investments are taken. We observe from the first order conditions of Problem (4) that for the optimal solution holds that

$$c_n^{\text{inv}} = \int_{t_0}^{t_e} \beta_{t,n}^+(\bar{y}) dt. \quad (16)$$

This means that only those scenarios contribute to the investment costs of a node, in which the node has variable costs that are strictly lower than the price in its price cluster, i.e., $\beta_{t,n}^+(\bar{y}) > 0$.

Combining these observations we see that the network structure induces investment incentives to install capacity close to consumers: The prices for nodes that consume in many scenarios are high and thus it is interesting to invest there. If the network exhibits a persistent bottleneck that manifests itself in most scenarios then investment in capacity on the demand side of that bottleneck will be efficient even if variable costs for the respective technology is higher.

The comparison to the classical peak-load pricing settings without a network is instructive. Despite the difference in the respective subproblems (without network the subproblem reduces to Problem (6) with a single cluster for all times t), the overall structure (16) of the investment solution is similar. If we have only one price cluster for all times (i.e., we have “no network”), the investment solution will strictly prefer nodes with low variable costs irrespective of their position in the network. This can lead to wildly different investment solutions and may especially lead to over- resp. underinvestment in the case of persistent bottlenecks.

This immediately suggests that zonal pricing could be used in order to solve the trade-off between the local distribution of capacity investments and network expansion. Indeed, several contributions that use the uniqueness result derived in this paper show that price clusters might adjust incentives in the right direction

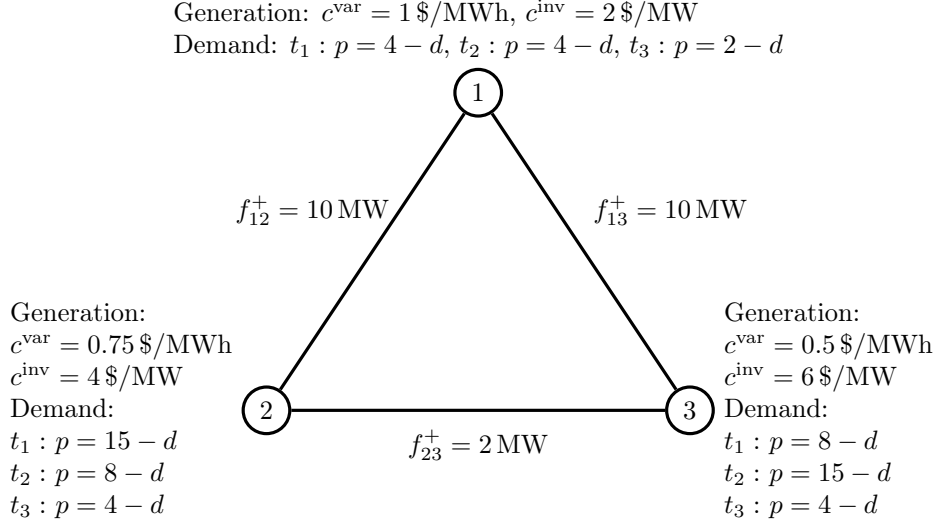


FIGURE 2. Three-Node Network

(see, e.g., Grimm et al. (2015) and the references therein). However, price clusters need to be configured carefully in order to achieve a welfare improvement.

We finally close this section with some technical remarks on the proven results. First, we remark that all results also apply to the situation in which multiple producers with different technologies are located at the nodes. This can be easily seen by introducing an auxiliary node for every producer at the node and by connecting the auxiliary nodes with the original nodes by arcs with “infinite” capacity. Second, the results are also valid for the case in which we replace the time continuous time horizon $T = [t_0, t_e] \subset \mathbb{R}$ with a discrete set of time periods $T = \{t_0, t_1, \dots, t_e\}$ with $t_i < t_{i+1}$. However, some of the assumption have to be adjusted accordingly.

At last, we want to mention that more complex flow models, such as a fully-fledged physical model as it is used to determine nodal prices, does not result in price clusters. Since our proofs heavily rely on the existence of these clusters, a proof of uniqueness of solutions of a peak-load pricing model on more complex networks requires completely different techniques. This is a topic for future research.

4. ILLUSTRATIVE EXAMPLE: THREE-NODE NETWORK

In this section we consider a three-node network that illustrates our concepts and theoretical results. The important features of this example are that the price clusters change over time and one scenario has non-unique flows. The changing price clusters can be directly observed in the structure of the Hessians corresponding to the different scenarios. As depicted in Fig. 2, the three nodes are connected by three arcs. At the three nodes investment in production capacity can take place with investment and production costs as shown in the figure. We consider three scenarios and linear demand functions that vary across these scenarios. The scenarios last 1 h and all data of the corresponding scenarios are constant during that time. Observe that demand at node 1 is relatively low in all scenarios. Scenario 1 (scenario 2) is characterized by the high (low) demand at node 2 and a comparatively low (high) demand at node 3. In scenario 3 overall demand is low.

Table 2 and 3 list the primal and dual solutions. The optimal solution shows that it is efficient to install 20 MW of new capacity at node 1, whereas capacity investment is not profitable at node 2 and 3. In scenario 1, arcs (1, 2) and (2, 3) are

TABLE 2. Primal Solution of the Three-Node Network

	d_1	d_2	d_3	p_1	p_2	p_3	y_1	y_2	y_3	f_{12}	f_{23}	f_{13}
Scenario 1	2	12	6	2	3	2	20	0	0	10	-2	8
Scenario 2	2	6	12	2	2	3	20	0	0	8	2	10
Scenario 3	1	3	3	1	1	1	7	0	0	3	0	3

TABLE 3. Dual Solution of the Three-Node Network

	α_1	α_2	α_3	β_1^+	β_2^+	β_3^+
Scenario 1	2	3	2	1	2.31	1.87
Scenario 2	2	2	3	1	1.31	2.85
Scenario 3	1	1	1	0	0.32	0.95

saturated. Therefore, in scenario 1 there are two different price clusters, which are formed by a flow-induced partition (see Definition 2): $C_{1,1} = \{2\}$ and $C_{1,2} = \{1, 3\}$. As can be seen in the tables, both prices (p) and dual variables of corresponding flow balance constraints (α) are identical for nodes 1 and 3 (see Definition 1). In analogy to scenario 1, in scenario 2 we also have two price clusters given the saturated lines (1, 3) and (2, 3). Thus, we have $C_{2,1} = \{3\}$ and $C_{2,2} = \{1, 2\}$.

Now consider the last scenario 3, in which no line is saturated. This yields a single price cluster $C_{3,1} = N$. It can be easily seen that the intra-cluster flows are not unique (see Theorem 2) since adding a small cycle flow is still feasible and does not change the objective function value.

To show that optimal capacity investment is unique, we next compute the Hessian of the master problem for the considered example. As stated in Proposition 3, the Hessian

$$H = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{3} & -\frac{4}{3} \end{bmatrix}$$

can be expressed as the sum of the Hessians H_t , $t = 1, 2, 3$:

$$H_1 = \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}, \quad H_2 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$$

The Hessian H is negative definite, i.e., Proposition 7 holds, and thus optimal production and capacity investment is unique (see Theorem 4).

5. CONCLUSION

In this paper we have analyzed a framework of peak-load pricing on a network where competitive firms take investment and production decisions facing network constraints expressed by fixed inter-zonal capacities. We have shown existence and uniqueness of the solution and characterized equilibrium investments. We also presented an approach that sheds light on a market where markups can be charged—although a full-fledged analysis of strategic interaction is not possible in our setup.

Our results are an important prerequisite for the analysis of energy policy proposals using multilevel computational equilibrium frameworks. These approaches can only be meaningfully used if lower-level problems have unique solutions that restrict feasible solutions at higher levels. This has been emphasized by various

authors, e.g., Dempe (2002), Colson et al. (2007), or Gabriel et al. (2012). Our contribution provides such a result for electricity market analyses. In Grimm et al. (2015) the result is already used in order to analyze optimal transmission expansion in liberalized electricity markets under different regulatory regimes.

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APPENDIX A. PROOF OF THEOREM 1

For completeness, we sketch a self-contained proof of Theorem 1, which is a variant of a result given in Schewe and Schmidt (2015). For this we need two observations: First, the max-flow-min-cut theorem immediately allows to us to write down an alternative characterization of the feasibility of Subproblem (5). This is a restatement of the well-known theorems of Gale and Hoffman for our particular case, see Schrijver (2003, Chapter 11).

Proposition 8. *The vector (d, y, f) is a feasible solution of Problem (5) if and only if $\sum_{n \in N} d_n = \sum_{n \in N} y_n$ and for all $C \subseteq N$, it holds that*

$$\sum_{n \in C} d_n - \sum_{n \in C} y_n \leq \hat{f}_C,$$

where $\hat{f}_C = \sum_{a \in \delta^{\text{in}}(C)} f_a^+ - \sum_{a \in \delta^{\text{out}}(C)} f_a^+$.

The second observation states that if two non-disjoint sets are tight at the upper bound, its union and intersection are as well. As a byproduct we obtain that in this case the connecting arcs between these two sets are unused. This result is also well-known, see again Schrijver (2003).

Proposition 9. *Let (d, y, f) be a feasible solution of Problem (5) and let $X, Y \subseteq N$ with $X \cap Y \neq \emptyset$ such that*

$$\sum_{n \in X} (d_n - y_n) = \hat{f}_X, \quad \sum_{n \in Y} (d_n - y_n) = \hat{f}_Y.$$

Then the following equations hold as well:

$$\sum_{n \in X \cap Y} (d_n - y_n) = \hat{f}_{X \cap Y}, \quad \sum_{n \in X \cup Y} (d_n - y_n) = \hat{f}_{X \cup Y}.$$

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Set $z^0 := (d^*, y^*)$ and let $z^1 = (d^1, y^1)$ be a solution of Problem (6). As z^0 is feasible for Problem (6), $\phi(z^0) \leq \phi(z^1)$ holds. It remains to show that $\phi(z^0) \geq \phi(z^1)$ holds. If there exists a flow f^1 such that (d^1, y^1, f^1) is feasible for Problem (5), $\phi(z^1) \leq \phi(z^0)$ holds and we obtain the desired inequality. Assume no such flow exists. Then it follows from Proposition 8 that there exists a set $U \subset N$ with $U \neq \emptyset$ such that

$$\sum_{n \in U} d_n^1 - \sum_{n \in U} y_n^1 > \hat{f}_U.$$

From the construction of \mathcal{C} it follows from Proposition 9 that we may assume that $U \subset C$ for a $C \in \mathcal{C}$. Set \mathcal{U} to be the set of all such sets U and define $z^\lambda := (1 - \lambda)z^0 + \lambda z^1$ for $\lambda \in [0, 1]$. It now follows that there exists a $\lambda_U > 0$ for each $U \in \mathcal{U}$ such that z^λ satisfies

$$\sum_{n \in U} d_n^\lambda - \sum_{n \in U} y_n^\lambda \leq \hat{f}_U.$$

Set $\rho := \min_{U \in \mathcal{U}} \lambda_U$. Then, it again follows from Proposition 8 that there exists a flow f such that (z^ρ, f) is feasible for Problem (5). Since $\rho > 0$, it also follows $\phi(z^\rho) > \phi(z^0)$, which is a contradiction to the optimality of z^* . Hence, $\phi(z^0) \geq \phi(z^1)$ holds. \square

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